Grandfather's clock

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Grandfather is building a new clock, a pendulum clock from the good old days. He is well aware that a gravitational restoring force will accelerate a sideways displaced and released pendulum towards its equilibrium position. As a consequence, the pendulum will display oscillations that can be used to measure time. Being an experienced craftsman, grandfather is planning his project in great detail. He is trying to find the oscillation period by modelling his clock mathematically as a point mass m suspended with a massless, rigid rod of length ℓ from a frictionless pivot. Grandfather is sure that the period of oscillation is somehow proportional to the initial displacement angle, i.e. the oscillation amplitude φ_0 measured from the equilibrium. His next-door neighbour, a retired engineer, thinks this task is redundant since it is common knowledge that the period is independent on the oscillation amplitude. Help grandfather to settle his dispute with the engineer.

(a) [**2 points**] Form the classical Hamiltonian function for the displaced pendulum.
(b) [**2 points**] Find the exact integral formula for the oscillation period by using the conservation of energy.
(c) [**2 points**] Show that in the limit of small initial displacement you can recover the engineer's result.
(d) [**2 points**] Finally, show grandfather's superiority over the engineer by calculating the leading order correction to the small displacement period obtained in **(c)**. At some point of your calculations, you might benefit from defining a quantity $\sin x \equiv \sin(\varphi/2)/\sin(\varphi_0/2)$, where φ is deviation angle from the equilibrium.

Grandfather is a well-educated man and knows that things in nature behave according to quantum laws, instead of the classical ones. He also tries to find out whether quantum mechanics brings any changes to the classical oscillation period. Figure out what grandfather will find.

(e) [**2 points**] Quantize the Hamiltonian function you obtained in **(a)** in the small oscillation limit. Find such linear combination $\hat{a}$ of your canonically conjugated variables that $[\hat{a}, \hat{a}^\dagger] = 1$ and $\hat{H} = \hbar\omega_p(\hat{a}^\dagger\hat{a} + \frac{1}{2})$, where ω_p is the classical small oscillation angular frequency. Notice that such $\hat{a}$ can be interpreted as the annihilation operator of the pendulum.
(f) [**2 points**] Consider first excited state of the pendulum in the small amplitude limit. One can approximate the classical oscillation amplitude φ_0 by equating the initial classical potential energy with the excited state energy where the zero-point fluctuations have been subtracted. Using this, find the leading order correction to the small amplitude oscillation period by using quantum mechanical perturbation theory. You can estimate the oscillation frequency by calculating the difference between the perturbed excited and ground state energies.